MULTIVARIABLE (φ,Γ)-MODULES AND REPRESENTATIONS OF PRODUCTS OF GALOIS GROUPS: THE CASE OF IMPERFECT RESIDUE FIELD

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Abstract. Let K be a complete discretely valued field with mixed characteristic $(0,p)$ and imperfect residue field k_α. Let Δ be a finite set. We construct an equivalence of categories between finite dimensional $\mathbb{F}_p$-representations of the product of Δ copies of the absolute Galois group of K and multivariable étale (φ,Γ)-modules over a multivariable Laurent series ring over k_α.

Résumé ((φ,Γ)-modules multivariables et représentations du produit du groupe de Galois: le cas des corps résiduels imparfaits)
Soit K un corps discrètement valué à charactéristique mixte $(0,p)$ et un corps résiduel imparfait k_α. Soit Δ un ensemble fini. Nous établissons une équivalence de catégories entre des représentations de dimensions finies sur $\mathbb{F}_p$ du produit de Δ copies du groupe absolu de Galois de K et des (φ,Γ)-modules étales multivariables sur un anneau multivariable des séries Laurent sur k_α.

1. Introduction

1.1. **Motivation of this work.** Fontaine's theory of (φ,Γ)-modules is a fundamental tool to describe and classify continuous representations of the Galois group of a finite extension of $\mathbb{Q}_p$ on a finite-dimensional $\mathbb{Q}_p$-vector space. With the help of Fontaine's theory of (φ,Γ)-modules, one can understand the p-adic and mod-p Langlands correspondence in the case of the general linear group GL_2 over the field $\mathbb{Q}_p$ of p-adic numbers, see [9–11, 13, 20–22, 41]. By invoking the theory of (φ,Γ)-modules, the p-adic and mod-p representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ can be connected with p-adic and mod-p Galois representations of $\mathbb{Q}_p$. To extend the correspondence to other p-adic reductive groups beyond $\mathrm{GL}_2(\mathbb{Q}_p)$, one naturally wants to generalize Fontaine's theory of (φ,Γ)-modules. There have been conjectural progress in attempts to generalize p-adic Langlands beyond $\mathrm{GL}_2(\mathbb{Q}_p)$ along these lines; two kinds of multivariable version of (φ,Γ)-modules can be found in the literature. Berger's multivariable (φ,Γ)-modules is an attempt to generalize p-adic Langlands for $\mathrm{GL}_2(F)$, where F is a finite extension of $\mathbb{Q}_p$ [6, 7]. The third author of this current work also defines multivariable (φ,Γ)-module over a m-variable Laurent series ring in an attempt to generalize p-adic Langlands for $\mathrm{GL}_m(\mathbb{Q}_p)$ [44, 49, 50]. One might also try to look at Zábrádi's multivariable (φ,Γ)-modules over Lubin-Tate extension to conjecturally understand p-adic Langlands for $\mathrm{GL}_m(F)$ [28]. It has become clear that essentially all of p-adic Hodge theory can be formulated in terms of

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(φ,Γ)-modules; moreover, this formulation has driven much recent progress in the subject and powered some notable applications in arithmetic geometry [17]. See [31] for a quick introduction to this circle of ideas or [43] for a more in-depth treatment. Multivariable (φ,Γ)-modules are also related [19, 32] to Scholze's theory of perfectoid spaces.

This paper can be considered as a complement to the third author's independent work [50] in which he shows that the category of continuous representations of the m^{th} direct product of the absolute Galois group of $\mathbb{Q}_p$ on finite dimensional $\mathbb{F}_p$-vector spaces (resp. $\mathbb{Z}_p$-modules, resp. $\mathbb{Q}_p$-vector spaces) is equivalent to the category of étale multivariable (φ,Γ)-modules over a certain m-variable Laurent series ring over $\mathbb{F}_p$ (resp. over $\mathbb{Z}_p$, resp. over $\mathbb{Q}_p$). In the current paper, we are going to extend this equivalence of categories for continuous $\mathbb{F}_p$-representations of the m^{th} direct product of the absolute Galois group of a complete discretely valued field K with mixed characteristic $(0,p)$ whose residue field k_α is imperfect and has a finite p-basis, i.e. $[k_\alpha : k_\alpha^p] = p^d$ (for some $d \geq 1$). We plan to come back to the question of p-adic representations in the future. We expect applications of our results to p-adic Hodge theory of products of varieties over p-adic fields. To state our main Theorem (Theorem 5.13) precisely, we need to review the third author's work on multivariable (φ,Γ)-modules [50] and his main theorem.

1.2. **Zábrádi's work [50].** Let F be a finite extension of $\mathbb{Q}_p$ with residue field k_F (which is perfect). For a finite set Δ, let $\mathcal{G}_{\mathbb{Q}_p,\Delta} := \prod_{\alpha\in\Delta} \mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ denote the direct power of the absolute Galois group of $\mathbb{Q}_p$ indexed by Δ. We denote by $\mathrm{Rep}_{k_F}(\mathcal{G}_{\mathbb{Q}_p,\Delta})$ the category of continuous representations of the profinite group $\mathcal{G}_{\mathbb{Q}_p,\Delta}$ on finite dimensional k_F-vector spaces. For independent commuting variables $X_\alpha\,(\alpha\in\Delta)$, we write

$$E_{\Delta,k_F} := k_F\,[\![X_\alpha|\alpha\in\Delta]\!]\,[X_\Delta^{-1}],$$

where $X_\Delta = \prod_{\alpha\in\Delta} X_\alpha$. For each element $\alpha\in\Delta$, we have the partial Frobenius φ_α, and group $G_{K_\alpha} \cong \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$ acting on the variable X_α in the usual way and commuting with the other variables $X_\beta\,(\beta\in\Delta\backslash\{\alpha\})$ in the ring E_{Δ,k_F} (some authors also write G_{K_α} as Γ_α). A $(\varphi_\Delta,\Gamma_\Delta)$-*module* (or a $(\varphi_\Delta,G_\Delta)$-module) over E_{Δ,k_F} is a finitely generated E_{Δ,k_F}-module D together with commuting semilinear actions of the operators φ_α and groups $G_{K_\alpha}\,(\alpha\in\Delta)$. We say that D is *étale* if the map $\mathrm{id}\otimes\varphi_\alpha : \varphi_\alpha^* D \longrightarrow D$ is an isomorphism for all $\alpha\in\Delta$. Then the third author independently shows that $\mathrm{Rep}_{k_F}(\mathcal{G}_{\mathbb{Q}_p,\Delta})$ is equivalent to the category of étale $(\varphi_\Delta,G_\Delta)$-modules over E_{Δ,k_F}.

1.3. **Andreatta's work [1] and Scholl's work [46].** Let us review Scholl's work [46] and parts of Andreatta's work [1] where they work with single variable classical (φ,Γ)-module but over an imperfect residue field. Let K be a complete discretely valued field (with uniformizer p) of mixed characteristic $(0,p)$ with imperfect residue field k_K having a p-basis, i.e. $[k_K : k_K^p] = p^d$. Let $t_1, t_2, \cdots, t_d \in K$ be a lift of a p-basis $\overline{t_1}, \overline{t_2}, \cdots, \overline{t_d}$ of k_K. Define $K_\infty = \bigcup_n K(\mu_{p^n}, t_1^{1/p^n}, \cdots, t_d^{1/p^n})$, $G_K = \mathrm{Gal}(K_\infty/K)$ and $\mathcal{G}_K = \mathrm{Gal}(\overline{K}/K)$. Note that, in contrast with the perfect residue field case, G_K is not abelian. Scholl [46] and Andreatta [1] defined a field of norms E_K for K, and have shown that $E_K \cong k_K((\overline{\pi}))$, where $\varepsilon = \overline{\pi}+1 \in E_K$ is compatible system of p-power roots of unity in K_∞ (cf. [46, Section 2.3]). Finally, Andreatta [1, Theorem 7.11] showed that $\mathrm{Rep}_{\mathbb{F}_p}(\mathcal{G}_K)$ is equivalent to the category of (single variable, i.e. classical) étale (φ, G_K)-mdule over E_K.

1.4. **Our work in this paper.** In this paper, we will extend Scholl's and Andreatta's result to the case of multivariable $(\varphi_\Delta, G_\Delta)$-modules over an imperfect residue field. Precisely speaking, for a finite set Δ and a collection of possibly distinct fields K_α as above, we define

$$G_\Delta = \prod_{\alpha\in\Delta} G_{K_\alpha},$$

$$\mathcal{G}_\Delta = \prod_{\alpha\in\Delta} \mathcal{G}_{K_\alpha},$$

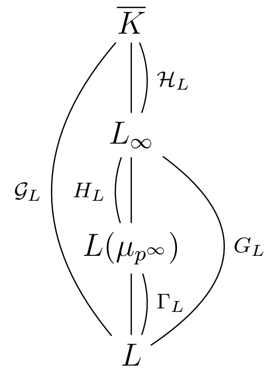
and the Laurent series ring

$$E_\Delta := (\bigotimes_{\alpha\in\Delta} k_{K_\alpha})\,[\![X_\alpha|\alpha\in\Delta]\!]\,[X_\Delta^{-1}].$$

It should be remarked that for each α, $G_{K_\alpha} \cong \Gamma_\alpha \ltimes H_\alpha$, where $\Gamma_\alpha \cong \mathrm{Gal}(K(\mu_{p^\infty})/K)$ and $H_\alpha \cong \mathrm{Gal}(K_\infty/K(\mu_{p^\infty}))$ and so G_{K_α} is a noncommutative p-adic Lie group. Extending actions of [46], we provide the ring E_Δ with the natural actions of partial Frobenius $\varphi_\alpha\,(\alpha\in\Delta)$, absolute Frobenius $\varphi_s = \prod_{\alpha\in\Delta}$ and the Galois group G_Δ. We define the category $\mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_\Delta, E_\Delta)$ of multivariable étale $(\varphi_\Delta, G_\Delta)$-modules over E_Δ in Section 3.3. Our main Theorem (see Theorem 5.13) is that there is an equivalence of categories between $\mathrm{Rep}_{\mathbb{F}_p}(\mathcal{G}_\Delta)$ and $\mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_\Delta, E_\Delta)$. Fortunately, many arguments in the proofs given by the third author in [50] (in the perfect residue field case) can be generalized and adapted to the case when the residue field is imperfect. Therefore, our proofs will mostly follow the line of arguments given in [50] with modifications when necessary, invoking the results of Andreatta and Scholl (for the single variable, imperfect residue field case) and using induction.

2. Kummer Towers

In this Section, we will introduce the Iwasawa theoretic tower that we are going to work with. Let L be a complete discretely valued field of mixed characteristic $(0,p)$. Suppose that $[k_L : k_L^p] = p^d$, where k_L is the residue field of L. Let us choose a complete subfield K of L with the same residue field k_L in which p is an uniformizer (the existence of such a subfield is proved in [34, Page 211-212]). Let $t_1, t_2, \cdots, t_d \in L$ be a lift of a p-basis $\overline{t_1}, \overline{t_2}, \cdots, \overline{t_d}$ of k_L. For $n \geq 1$, define $K_n = K(\mu_{p^n}, t_1^{1/p^n}, \cdots, t_d^{1/p^n})$, $K_\infty = \bigcup_n K_n$, $L_n = LK_n$ and $L_\infty = LK_\infty$. Define the Galois groups $\Gamma_L = \mathrm{Gal}(L(\mu_{p^\infty})/L)$, $G_L = \mathrm{Gal}(L_\infty/L)$, $H_L = \mathrm{Gal}(L_\infty/L(\mu_{p^\infty}))$, $\mathcal{H}_L = \mathrm{Gal}(\overline{K}/L_\infty)$, $\mathcal{G}_L = \mathrm{Gal}(\overline{K}/L)$. We identify Γ_L via the quotient map with the subgroup $\mathrm{Gal}(L_\infty/L'_\infty)$ of G_L, where $L_\infty = \varinjlim_n L(t_1^{1/p^n}, \cdots, t_d^{1/p^n})$.



Note that the cyclotomic character χ identifies Γ_L with an open subgroup of $\mathbb{Z}_p^\times$. We also have that $G_L \cong \Gamma_L \ltimes H_L$, where $H_L \cong \mathbb{Z}_p^d$ and G_L is a non-commutative p-adic Lie group of dimension $d+1$.The tower $(K_n)_{n\geq 1}$ is strictly deeply ramified in the sense of [46]. By [46, Section 1.3], we can say that there exists $n_0 \in \mathbb{N}$ and $\xi \in \mathcal{O}_{K_{n_0}}$ satisfying $0 < |\xi|_K < 1$ such that for all $n \geq n_0$, the p-power map $\mathcal{O}_{K_{n+1}}/(\xi) \longrightarrow \mathcal{O}_{K_n}/(\xi)$ is a surjection. We denote $E_K^+ = \varprojlim_{n\geq n_0} \mathcal{O}_{K_n}/(\xi)$, where the inverse limit is taken with respect to the p-power maps. Then E_K^+ is a complete, discretely valued ring of characteristic p, independent of n_0 and ξ (cf. [51, Section 2.1]). Let E_K be the fraction field of E_K^+. We call E_K the *field of norms* of the tower K_n. Note that E_K has a natural action of G_K that commutes with the Frobenius operator φ.

For every finite extension K' of K, $E_{K'}$ is a finite separable extension of E_K. Let $E_K^{\mathrm{sep}} = \bigcup_{K'} E_{K'}$. It follows from [1, Corollary 6.4] that there is an isomorphism of topological groups

$$\mathrm{Gal}(E_K^{\mathrm{sep}}/E_K) \cong \mathrm{Gal}(\overline{K}/K_\infty) = \mathcal{H}_K.$$

We therefore conclude that $(E_K^{\mathrm{sep}})^{\mathcal{H}_K} \cong E_K$.

Theorem 2.1. [46, Section 2.3] *The field of norms $E_K \cong k_K((\overline{\pi}))$, where $\varepsilon = 1 + \overline{\pi}$ is a compatible system of p-power roots of unity.*

The field of norms $E_L = (E_K^{sep})^{\mathcal{H}_L}$ for the tower L_∞ is a finite separable extension of E_K of the form $E_L \cong k_K((X))$ for some noncanonical choice of uniformizer X. Therefore the action of G_L on E_L does not have an intrinsic description in general. We shall further make the following hypothesis on K (hence on L, as we have $k_K = k_L$).

HYP 2.2. *The residue field k_K is a finitely generated field extension of $\mathbb{F}_p$.*

Let Δ be a finite set and we pick a complete discretely valued field $L = L_\alpha$ as above for each α. We also allow the cardinality d_α of a p-basis to vary for $\alpha \in \Delta$. We put $k_\alpha := k_{K_\alpha} = k_{L_\alpha}$ and assume that it satisfies HYP 2.2 for all $\alpha \in \Delta$. Further, let $\mathbb{F}_\alpha$ denote the maximal algebraic extension of $\mathbb{F}_p$ contained in k_α. Then $\mathbb{F}_\alpha = k_\alpha((X_\alpha))^{G_{L,\Delta}}$ is a finite field and k_α is a finite separable extension of the function field $k_{\alpha,0} := \mathbb{F}_\alpha(\overline{t_{\alpha,1}}, \ldots, \overline{t_{\alpha,1}})$ where $\overline{t_{\alpha,1}}, \ldots, \overline{t_{\alpha,1}} \in k_\alpha$ is a finite p-basis. We put $\mathbb{F}_\Delta := \bigotimes_{\mathbb{F}_p,\alpha\in\Delta} \mathbb{F}_\alpha$ and $k_{\Delta,0} := \bigotimes_{\mathbb{F}_p,\alpha\in\Delta} k_{\alpha,0}$.

Lemma 2.3. *Assume HXP 2.2 for each residue field k_α. Then the $|\Delta|$-fold tensor product $k_\Delta := \bigotimes_{\mathbb{F}_p,\alpha\in\Delta} k_\alpha$ is noetherian and regular (and, in particular, reduced). Further, for each $\alpha \in \Delta$ the relative Frobenius φ_α is injective on k_Δ.*

Proof. The ring k_Δ is the localization of a finitely generated $\mathbb{F}_p$-algebra, therefore it is noetherian. Since $\mathbb{F}_p$ is a field, any $\mathbb{F}_p$-module is flat. Now $\varphi_\alpha\colon k_\alpha\to k_\alpha$ is injective and on $k_{\Delta\setminus\{\alpha\}}$ it is the identity, therefore φ_α is also injective on $k_\Delta=k_\alpha\otimes_{\mathbb{F}_p}k_{\Delta\setminus\{\alpha\}}$. In particular, the absolute Frobenius $\varphi_s=\prod_{\alpha\in\Delta}\varphi_\alpha$ is also injective on k_Δ, ie. k_Δ is reduced. The statement on the regularity follows from [47, Theorem 1.6(c),(e)] since $\mathbb{F}_p$ is perfect and k_Δ is noetherian. □

Since $\mathbb{F}_\Delta$ is a tensor product of finite fields, it is finite étale algebra over $\mathbb{F}_p$. Moreover, it has primitive idempotents $b_1,\dots,b_\ell\in\mathbb{F}_\Delta$ with $1=b_1+\dots+b_\ell$ and $b_j\mathbb{F}_\Delta\cong\mathbb{F}_{p^f}$ where $f=\gcd(|\mathbb{F}_\alpha:\mathbb{F}_p|\mid\alpha\in\Delta)$, $1\le j\le\ell$. Note that for each $\alpha\in\Delta$ and $1\le j\le\ell$ the element $\varphi_\alpha(b_j)$ is also a primitive idempotent in $\mathbb{F}_\Delta$ and $\varphi_s(b_j)=b_j^p=b_j$. The quotient monoid $\Phi:=(\prod_{\alpha\in\Delta}\varphi_\alpha^{\mathbb{N}})/\varphi_s^{\mathbb{N}}$ is a group acting on the set of primite idempotents.

Lemma 2.4. *The group Φ acts transitively on the set $\{b_1,\dots,b_\ell\}$ of primitive idempotents in $\mathbb{F}_\Delta$.*

Proof. By induction on $|\Delta|$ we are reduced to the case $\Delta=\{\alpha,\beta\}$ has two elements. Put $p^n:=|\mathbb{F}_\alpha|$, $p^m:=|\mathbb{F}_\beta|$, and $f:=\gcd(n,m)$. Writing $\mathbb{F}_{p^m}=\mathbb{F}_p[X]/(g(X))$ for some irreducible monic polynomial $g(X)\in\mathbb{F}_p[X]$ we may write $g(X)=\prod_{i=1}^f g_i(X)$ over $\mathbb{F}_{p^n}$ so we have $\mathbb{F}_{p^n}\otimes_{\mathbb{F}_p}\mathbb{F}_{p^m}=\bigoplus_{i=1}^f\mathbb{F}_{p^n}[X]/(g_i(X))$ where φ_β acts trivially on $\mathbb{F}_{p^n}$ and satisfies $\varphi_\beta(g_i)=g_{i+1}$ with the convention $g_{f+1}=g_1$. □

3. Multivariable (φ,Γ)-modules

3.1. Let Δ be a finite set (which can be simple roots in the Lie algebra of a reductive group over $\mathbb{Z}_p$) and $(L_\alpha)_{\alpha\in\Delta}$ be a collection of complete discretely valued fields with residue fields k_α such that k_α is a finitely generated extension of $\mathbb{F}_p$ with finite p-basis $\overline{t_{\alpha,1}},\dots,\overline{t_{\alpha,1}}$. We further choose a complete subfield $K_\alpha\le L_\alpha$ in which p is a uniformizer and has the same residue field k_α. Let us define

i) $G_{L,\Delta}:=\prod\limits_{\alpha\in\Delta}G_{L_\alpha}$,
ii) $\mathcal{G}_{L,\Delta}:=\prod\limits_{\alpha\in\Delta}\mathcal{G}_{L_\alpha}$.

We denote $\mathrm{Rep}_{\mathbb{F}_p}(\mathcal{G}_{L,\Delta})$ the category of continuous representations of the profinite group $\mathcal{G}_{L,\Delta}$ on finite dimensional $\mathbb{F}_p$-vector spaces. (In the future, $G_{K,\Delta}$ and $\mathcal{G}_{K,\Delta}$ will simply be denoted as G_Δ and $\mathcal{G}_\Delta$, dropping the subscript K.)

3.2. **Some Laurent series rings.** Consider the Laurent series $E_{L,\Delta}=E_{L,\Delta}^+[X_\Delta^{-1}]$ where

$$E_{L,\Delta}^+=k_\Delta[\![X_\alpha|\alpha\in\Delta]\!]=(\bigotimes_{\mathbb{F}_p,\alpha\in\Delta}k_\alpha)[\![X_\alpha|\alpha\in\Delta]\!]$$

is the completed tensor product of E_α^+ ($\cong k_\alpha[\![X_\alpha]\!]$) over $\mathbb{F}_p$ for all $\alpha\in\Delta$. Here E_α^+ is the ring of integers of the field of norms E_α ($\cong k_\alpha((X_\alpha))$) corresponding to α. Here $X_\Delta:=\prod\limits_{\alpha\in\Delta}X_\alpha\in E_\Delta^+$. For each α, we define the action of the partial Frobenius φ_α and the group G_{L_α} as follows (cf. [46, Page 707, Section 2.3]).

$$\varphi_\alpha(X_\beta) := \begin{cases} X_\beta & \text{if } \beta \in \Delta\backslash\{\alpha\}, \\ (1+X_\alpha)^p - 1 = X_\alpha^p & \text{if } \beta = \alpha. \end{cases} \tag{3.2.1}$$

φ_α acts on the coefficients in k_α as the p-th power map and as identity on k_β for $\beta \in \Delta \setminus \{\alpha\}$. By the construction of the field of norms [46], the group G_{L_α} acts on the ring $k_\alpha((X_\alpha))$ and we extend this action to $E_{L,\Delta}$ by acting trivially on $k_\beta((X_\beta))$ $(\beta \in \Delta \setminus \{\alpha\})$. We only describe the action of G_{L_α} on $E_{L,\Delta}$ intrinsically in case $L_\alpha = K_\alpha$. We put $\overline{\pi}_\alpha := \varepsilon - 1 \in k_\alpha((X_\alpha)) \subset \varprojlim \mathcal{O}_{L_n}/(\xi)$ where $\varepsilon = (\varepsilon_n)_{n\geq 0}$ is a compatible system of p-power roots of unity (with $\varepsilon_0 = 1 \neq \varepsilon_1$). In case $L_\alpha = K_\alpha$ is unramified, we have $k_\alpha((X_\alpha)) = k_\alpha((\overline{\pi}_\alpha))$, but in general $k_\alpha((X_\alpha))$ is a finite separable extension of $k_\alpha((\overline{\pi}_\alpha))$ of degree e_L which is the absolute ramification index of L. Action of $G_{K_\alpha} = G_\alpha \cong \Gamma_\alpha \ltimes H_\alpha$; $H_\alpha \cong \mathbb{Z}_p^d$ on $E_\Delta := E_{K,\Delta}$ can be described as follows. For any $\gamma_\alpha \in \Gamma_\alpha$ we have

$$\gamma_\alpha(\overline{\pi}_\beta) := \begin{cases} \overline{\pi}_\beta & \text{if } \beta \in \Delta\backslash\{\alpha\}, \\ (1+\overline{\pi}_\alpha)^{\chi(\gamma_\alpha)} - 1 & \text{if } \beta = \alpha. \end{cases} \tag{3.2.2}$$

Γ_α acts as identity on k_α. Let $\underline{b}_\alpha$ be the image of $\delta_{\alpha,\underline{b}} \in H_\alpha$ in $\mathbb{Z}_p^d$ and let $b_{\alpha,i}$ be the i-th component of $\underline{b}_\alpha$. Then, we have

$$\delta_{\alpha,\underline{b}}(\overline{\pi}_\beta) := \overline{\pi}_\beta \text{ for all } \beta \text{ (equal to } \alpha \text{ and also not equal to } \alpha), \tag{3.2.3}$$

$$\delta_{\alpha,\underline{b}}(\overline{t_{\alpha,i}}) := (1+\overline{\pi}_\alpha)^{b_{\alpha,i}}\overline{t_{\alpha,i}} \text{ for all } 1 \leq i \leq d_\alpha , \tag{3.2.4}$$

and $\delta_{\alpha,\underline{b}}$ acts as identity on k_β for $\beta \neq \alpha$. Note that such an automorphism $\delta_{\alpha,\underline{b}}$ of E_Δ^+ is unique which is easy to see from the case $|\Delta| = 1$ in [46, Page 707].

Note that the absolute Frobenius $\varphi_s : E_{L,\Delta}^+ \longrightarrow E_{L,\Delta}^+$ equals the composite $\prod_{\alpha\in\Delta} \varphi_\alpha$ of the partial Frobenii. Further, the actions of φ_α $(\alpha \in \Delta)$ and G_β $(\beta \in \Delta)$ all commute with each other (even though the individual factors G_β are non-abelian).

The ring $E_{L,\Delta}^+$ is noetherian and reduced by Lemma 2.3.

3.3. **Multivariable $(\varphi_\Delta, G_{L,\Delta})$-modules.** By a $(\varphi_\Delta, G_{L,\Delta})$-module over $E_{L,\Delta}$, we mean a finitely generated module D over $E_{L,\Delta}$ together with commuting semilinear actions of φ_α and the Galois groups G_{L_α} for all $\alpha \in \Delta$. By an étale $(\varphi_\Delta, G_{L,\Delta})$-module over $E_{L,\Delta}$, we mean a $(\varphi_\Delta, G_{L,\Delta})$-module D such that the maps

$$\mathrm{id} \otimes \varphi_\alpha : \varphi_\alpha^* D := E_{L,\Delta} \bigotimes_{E_{L,\Delta},\varphi_\alpha} D \longrightarrow D,$$

are isomorphisms for all $\alpha \in \Delta$.

We are going to show that $\mathrm{Rep}_{\mathbb{F}_p}(\mathcal{G}_{L,\Delta})$ is equivalent to the category of étale $(\varphi_\Delta, G_{L,\Delta})$-modules over $E_{L,\Delta}$; the later category we denote by $\mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$.

4. Integrality Properties

4.1. **Definition and projectivity.** In this Section our goal is to show that any object in the category $\mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$ is stably free as a module over $E_{L,\Delta}$.

Lemma 4.1. *There exists a $G_{L,\Delta}$-equivariant injective resolution of $E_{L,\Delta}$ as a module over itself.*

Proof. This follows from a general result that the Cousin complex provides an injective resolution for spectrum of noetherian rings with finite injective dimension (these are the so-called Gorenstein rings, we recommend the reader to refer to [29, Remark before Proposition 3.4 in Page 249] or [42]). Note that $E_{L,\Delta}$ is the localization of a $|\Delta|$-variable power series ring over the regular ring k_Δ, therefore it is regular (and in particular, Gorenstein). Hence the Cousin complex

$$0 \longrightarrow E_{L,\Delta} \xrightarrow[d_{-1}]{} \bigoplus_{\mathfrak{p}\in\mathrm{Spec}(E_{L,\Delta}),\mathrm{ht}\mathfrak{p}=0} (E_{L,\Delta})_{\mathfrak{p}} \xrightarrow[d_0]{} \cdots \longrightarrow \bigoplus_{\mathfrak{p}\in\mathrm{Spec}(E_{L,\Delta}),\mathrm{ht}\mathfrak{p}=r} \mathrm{Coker}(d_{r-2})_{\mathfrak{p}} \xrightarrow[d_r]{} \cdots$$

is an injective resolution of $E_{L,\Delta}$. This resolution is $G_{L,\Delta}$-equivariant because the automorphisms preserve the height of a prime ideal. □

Recall $\mathbb{F}_\Delta$ has primitive idempotents $b_1,\dots,b_\ell \in \mathbb{F}_\Delta$ with $1 = b_1+\cdots+b_\ell$ and $b_j\mathbb{F}_\Delta \cong \mathbb{F}_{p^f}$ where $f = \gcd(|\mathbb{F}_\alpha : \mathbb{F}_p| \mid \alpha\in\Delta)$, $1\le j\le \ell$.

Lemma 4.2. *Let I be a $G_{L,\Delta}$-invariant ideal of $E_{L,\Delta}$. Then we have $I = (I\cap\mathbb{F}_\Delta)E_{L,\Delta}$.*

Proof. Suppose that I is a nontrivial $G_{L,\Delta}$-invariant ideal of $E_{L,\Delta}$. Then I is also Γ_Δ-invariant. We first show $I = (I\cap k_\Delta)E_{L,\Delta}$. This is completely analogous to the proof of [49, Proposition 2.1, Lemma 2.2]. The assumption that the ring κ of loc. cit. is a finite field is not used in the proof of [49, Lemma 2.2]: one only needs that $E_{L,\Delta}$ is noetherian. Further, t can be chosen large enough to ensure $1+p^t$ lies in the image of the character $\chi\colon \Gamma_{L_\alpha}\to\mathbb{Z}_p^\times$ for all $\alpha\in\Delta$. In our argument κ is not a field, but the ring k_Δ in which case one cannot conclude at the end of the proof of Proposition 2.1 that $1\in I$ but only that $I=(I\cap k_\Delta)E_{L,\Delta}$.

We use the action of $H_{L,\Delta}$ in order to further descend to $\mathbb{F}_\Delta$. Fix $\alpha\in\Delta$, $j\in\{1,\dots,\ell\}$, and pick an element $0\neq\lambda=\sum_{i=1}^r u_i\otimes v_i \in b_jI\cap k_\Delta$ where $u_1,\dots,u_r\in k_\alpha$ and $v_1,\dots,v_r\in k_{\Delta\setminus\{\alpha\}}$. Suppose r is minimal and $u_1=1$ (possibly replacing λ with $u_1^{-1}\lambda$), i.e. no nonzero element in $b_jI\cap k_\Delta$ can be written as a sum of at most $r-1$ elementary tensors. In particular, both $u_1,\dots,u_r$ and $v_1,\dots,v_r$ linearly independent over $\mathbb{F}_p$. Assume for contradiction that there is an index $i_0\in\{2,\dots,r\}$ such that $u_{i_0}\notin\mathbb{F}_\alpha$. Then there exists an element $h\in H_{L_\alpha}$ such that $h(u_{i_0})\neq u_{i_0}$ whence $0\neq\sum_{i=2}^r(h(u_i)-u_i)\otimes v_i = h(\lambda)-\lambda\in b_jI$. Writing $h(\lambda)-\lambda=\sum_{N=0}^\infty \lambda_N X_\alpha^N$ where $\lambda_N\in b_jI\cap k_\Delta$ can be written as a sum of at most $r-1$ elementary tensors for all $N\ge 0$ contradicts to the minimality of r. By repreating this argument for all $\alpha\in\Delta$ we find a nonzero element in $b_jI\cap\mathbb{F}_\Delta$ whenever $b_jI\neq 0$. Hence we deduce $b_jI = b_jE_{L,\Delta}$ as $b_j\mathbb{F}_\Delta$ is a field. The claim follows noting that j is arbitrary and $I=\bigoplus_{j=1}^\ell b_jI$. □

Lemma 4.3. *Any object D in $\mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$ is a projective module over $E_{L,\Delta}$.*

Proof. The proof follows from the argument in [50, Proposition 2.2] with the following modification: since $E_{L,\Delta}$ is not a domain, the assertion that $\mathrm{Ext}^n_{E_{L,\Delta}}(D,E_{L,\Delta})=0$ does not make sense. However, $\mathrm{Ext}^n_{E_{L,\Delta}}(D,E_{L,\Delta})=\bigoplus_{j=1}^\ell \mathrm{Ext}^n_{b_jE_{L,\Delta}}(b_jD,b_jE_{L,\Delta})$ is a finitely generated torsion $b_jE_{L,\Delta}$-module ($b_jE_{L,\Delta}$ is a domain). So we deduce by the proof of [50, Proposition 2.2] that b_jD is a projective $E_{L,\Delta}$-module for each $j=1,\dots,\ell$ whence so is $D=\bigoplus_{j=1}^\ell b_jD$. □

Lemma 4.4. *We have $K_0(b_jE_{L,\Delta}) \cong \mathbb{Z}$ for all $j = 1, \dots, \ell$, ie. any finitely generated projective module over $b_jE_{L,\Delta}$ is stably free.*

Proof. The proof was given in [50, Lemma 2.3]. □

Proposition 4.5. *Let D be an object in the category $\mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$. Then D is stably free as a module over $E_{L,\Delta}$.*

Proof. By Lemma 4.4 it remains to show that the rank of b_jD does not depend on j ($j = 1, \dots, \ell$). However, this follows from Lemma 2.4 as we have $E_{L,\Delta}\varphi_\alpha(b_jD) = \varphi_\alpha(b_j)D$ for all $\alpha \in \Delta$. □

4.2. **Topology of $E^+_{L,\Delta}$ and $E_{L,\Delta}$.** We equip $E^+_{L,\Delta}$ with the X_Δ-adic topology, and equip $E_{L,\Delta}$ with the inductive limit topology $E_{L,\Delta} = \bigcup_n X_\Delta^{-n}E^+_{L,\Delta}$. This makes $(E_{L,\Delta}, E^+_{L,\Delta})$ a Huber pair in the sense of [45]. $E_{L,\Delta}$ is a complete noetherian Tate ring (loc. cit.). Note that this is not the natural compact topology on $E^+_{L,\Delta}$ as in the compact topology $E^+_{L,\Delta}$ would not be open in $E_{L,\Delta}$ since the index of $E^+_{L,\Delta}$ in $X_\Delta^{-n}E^+_{L,\Delta}$ is not finite. Also, the inclusion $k_\alpha((X_\alpha)) \hookrightarrow E_{L,\Delta}$ is not continuous in the X_Δ-adic topology (unless $|\Delta| = 1$).

Suppose $D \in \mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$. By Banach's Theorem for Tate rings ([48, Proposition 6.18]), there is a unique $E_{L,\Delta}$-module topology on D that we call the X_Δ-*adic topology*. (This is the induced topology as D is finitely generated over $E_{L,\Delta}$). Moreover, any $E_{L,\Delta}$-module homomorphism is continuous in the X_Δ-adic topology.

Let M be a finitely generated $E^+_{L,\Delta}$-submodule in $D \in \mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$. Suppose that $\{m_1, m_2, \cdots, m_n\}$ is a set of generators of M. Then $\varphi_s(m_1), \cdots, \varphi_s(m_n)$ generate $E^+_{L,\Delta}\varphi_s(M)$. Thus $E^+_{L,\Delta}\varphi_s(M)$ is also finitely generated.

Now, let $D^{++} := \left\{x \in D \,\middle|\, \lim_{k\to\infty} \varphi_s^k(x) = 0\right\}$ where the limit is considered in the X_Δ-adic topology (cf. [20, II.2.1] in case $|\Delta| = 1$).

Proposition 4.6. *D^{++} is a finitely generated $E^+_{L,\Delta}$-submodule in D which is stable under the actions of $\varphi_\alpha, G_{L_\alpha}$ for all $\alpha \in \Delta$ and we have $D = D^{++}[X_\Delta^{-1}]$.*

Proof. The proof is essentially the same as [50, Proposition 2.5], but we would like to clarify few steps as we sketch the third author's line of proof.

Choose an arbitrary finitely generated $E^+_{L,\Delta}$-submodule M of D with $M[X_\Delta^{-1}] = D$. We can take $M = E^+_{L,\Delta}e_1 + \cdots + E^+_{L,\Delta}e_n$ for some $E_{L,\Delta}$-generating system $e_1, \cdots, e_n$ of D. First, note that M is not φ_s-stable, but $E^+_\Delta\varphi_s(M)$ is finitely generated (as M is finitely generated over E^+_Δ). Hence we can find a "common denominator" of $E^+_{L,\Delta}\varphi_s(M)$ to be X_Δ^r such that $\varphi_s(M) \subseteq X_\Delta^{-r}M$, since $E^+_{L,\Delta}$ is noetherian and we have $D = \bigcup_r X_\Delta^{-r}M$. Then we have

$$\varphi_s(X_\Delta^k M) = X_\Delta^{pk}\varphi_s(M) \subseteq X_\Delta^{pk-r}M \subseteq X_\Delta^{k+1}M$$

for any integer $k \geq \frac{r+1}{p-1}$. We therefore have $X_\Delta^{[\frac{r+1}{p-1}]+1}M \subseteq D^{++}$. This implies that

$$M[X_\Delta^{-1}] = D = X_\Delta^{[\frac{r+1}{p-1}]+1}M[X_\Delta^{-1}] \subseteq D^{++}[X_\Delta^{-1}].$$

But $D^{++}[X_\Delta^{-1}] \subseteq D$ is obvious. Thus $D^{++}[X_\Delta^{-1}] = D$. Note that D^{++} is stable under G_{L_α}, because the action of G_{L_α} commute with φ_s (and also φ_α for all $\alpha \in \Delta$). There is a system of neighbourhoods of 0 in D consisting of $E^+_{L,\Delta}$-submodules. And hence D^{++} is an $E^+_{L,\Delta}$-submodule.

Assume first D is a free module over $E_{L,\Delta}$ with free generators $e_1, \cdots, e_n$ and put $M := E^+_{L,\Delta}e_1 + \cdots + E^+_{L,\Delta}e_n$. Then we can show that $D^{++} \subseteq X_\Delta^{-r}M$ for some integer $r > 0$ (cf. [50, Proposition 2.5]). As $E^+_{L,\Delta}$ is noetherian, this gives that D^{++} is finitely generated over $E^+_{L,\Delta}$.

In the general case, by Proposition 4.6, we know that D is stably free. Therefore we can have $D_1 := D \bigoplus E^k_{L,\Delta}$ making D_1 into an étale free module over $(\varphi_\alpha, \varphi_s, G_{L_\alpha}, \alpha \in \Delta)$ by the trivial action of $(\varphi_\alpha, \varphi_s, G_{L_\alpha}, \alpha \in \Delta)$ on $E^k_{L,\Delta}$. This gives us that D_1^{++} is finitely generated over $E^+_{L,\Delta}$. The result follows as $D^{++} \subseteq D_1^{++}$ and $E^+_{L,\Delta}$ is noetherian. □

Let us define
$$D^+ := \{x \in D \,|\, \{\varphi_s^k(x) : k \geq 0\} \subset D \text{ is bounded}\}.$$
Since $\varphi_s^k(X_\Delta)$ tends to 0 in the X_Δ-adic topology, we have $X_\Delta D^+ \subseteq D^{++}$, i.e. $D^+ \subseteq X_\Delta^{-1}D^{++}$. In particular, D^+ is finitely generated over $E^+_{L,\Delta}$. On the other hand, we also have $D^{++} \subseteq D^+$ by construction whence we deduce $D = D^+[X_\Delta^{-1}]$.

Lemma 4.7. *For all $\alpha \in \Delta$ and $g_\alpha \in G_{L_\alpha}$ we have*
$$\varphi_\alpha(D^+) \subset D^+ \quad (\text{resp. } \varphi_\alpha(D^{++}) \subset D^{++}),$$
$$g_\alpha(D^+) \subset D^+ \quad (\text{resp. } g_\alpha(D^{++}) \subset D^{++}).$$

Proof. We will show that, for any generating system $e_1, \cdots, e_n$ of D and any γ (γ can be φ_α or $g_\alpha \in G_{L_\alpha}$), there exists an integer $k > 0$ such that
$$\gamma(X_\Delta^k M) \subseteq X_\Delta^k E^+_{L,\Delta}\gamma(M) \subseteq M,$$
where $M := E^+_{L,\Delta}e_1 + \cdots + E^+_{L,\Delta}e_n$.

Case (i) Assume that $\gamma = \varphi_\alpha$. Then choose $k \gg 0$ so that $X_\Delta^k E^+_{L,\Delta}\varphi_\alpha(M) \subseteq M$ whence
$$\varphi_\alpha(X_\Delta^k M) = \prod_{\beta \neq \alpha} X_\beta^k X_\alpha^{pk}\varphi_\alpha(M) = X_\Delta^k X_\alpha^{(p-1)k}\varphi_\alpha(M) \subseteq M\ .$$

Case (ii) Assume that $\gamma = g_\alpha \in G_{L_\alpha}$. We need to check that $g_\alpha(X_\Delta) = uX_\Delta$ for some unit $u \in E^+_{L,\Delta}$. In case $K = L$ this is clear from the intrinsic description of the action of G_{L_α}. The general statement follows noting that we still have $g_\alpha(X_\beta) = X_\beta$ for $\beta \neq \alpha$ and $g_\alpha(X_\alpha)$ is also a uniformizer in $k_\alpha((X_\alpha))$ since $k_\alpha((X_\alpha))$ is a finite separable extension of $k_\alpha((\overline{\pi}_\alpha))$ and G_{L_α} is a subgroup in G_{K_α}. The proof then follows from [50, Lemma 2.6]. □

We now fix an $\alpha \in \Delta$ and define $D^+_{\overline{\alpha}} := D^+[X^{-1}_{\Delta\setminus\{\alpha\}}]$ where for any subset $S \subseteq \Delta$ we put $X_S := \prod_{\beta \in S} X_\beta$. Then $D^+_{\overline{\alpha}}$ is a finitely generated module over $E^+_{\overline{\alpha}} := E^+_{L,\Delta}[X^{-1}_{\Delta\setminus\{\alpha\}}]$.

Lemma 4.8. *$D^+_{\overline{\alpha}}/D^+$ is X_α-torsion free: If both $X_\alpha^{n_1}d$ and $X^{n_2}_{\Delta\setminus\{\alpha\}}d$ lie in D^+ for some element $d \in D$, $\alpha \in \Delta$, and integers $n_1, n_2 \geq 0$ then we have $d \in D^+$. The same statement holds if we replace D^+ by D^{++}.*

Proof. The proof of [50, Lemma 2.7] works without any change. □

Lemma 4.9. *Assume that D is generated by a single element $e_1 \in D$ over $E_{L,\Delta}$. Then for any γ, we have $\gamma(e_1) = a_\gamma e_1$ for some unit $a_\gamma \in (E^+_{\overline{\alpha}})^\times$. Here γ can be $\varphi_\beta, g_\beta \in G_{L_\beta}$ for $\beta \neq \alpha$.*

Proof. For any γ equals to either g_β or φ_β, we define a_γ and a_α such that

$$\gamma(e_1) = a_\gamma e_1 \text{ and } \varphi_\alpha(e_1) = a_\alpha e_1.$$

By the étaleness property, it follows that D is generated by $\gamma(D)$ over $E_{L,\Delta}$. Thus $e_1 \in D$ implies

$$\begin{aligned} e_1 &= x\gamma(e_1) \quad (\text{for some } x \in E_{L,\Delta}) \\ &= xa_\gamma e_1 \\ &= y\varphi_\alpha(e_1) \quad (\text{for some } y \in E_{L,\Delta}, \text{as } D \text{ is also generated by } \varphi_\alpha(D) \text{ over } E_{L,\Delta}) \\ &= ya_\alpha e_1 \end{aligned}$$

So $xa_\gamma = ya_\alpha = 1$, which implies that both a_γ and a_α are units in $E_{L,\Delta}$. It remains to show that $\mathrm{val}_{X_\alpha}(a_\gamma) = 0$. We compute

$$\begin{aligned} \varphi_\alpha(a_\gamma)a_\alpha e_1 &= \varphi_\alpha(a_\gamma)\varphi_\alpha(e_1) = \varphi_\alpha(a_\gamma e_1) = \varphi_\alpha(\gamma(e_1)) \\ &= \gamma(\varphi_\alpha(e_1)) = \gamma(a_\alpha e_1) = \gamma(a_\alpha)\gamma(e_1) = \gamma(a_\alpha)a_\gamma e_1. \end{aligned}$$

And hence we deduce

$$p\mathrm{val}_{X_\alpha}(a_\gamma) + \mathrm{val}_{X_\alpha}(a_\alpha) = \mathrm{val}_{X_\alpha}(\varphi_\alpha(a_\gamma)a_\alpha) = \mathrm{val}_{X_\alpha}(\gamma(a_\alpha)a_\gamma). \tag{4.8.1}$$

Since φ_β and g_β act trivially on $k_\alpha((X_\alpha))$ and they are injective on $E_{L,\Delta}$, they both preserve the X_α-adic valuation. So we have

$$\mathrm{val}_{X_\alpha}(\gamma(a_\alpha)a_\gamma) = \mathrm{val}_{X_\alpha}(a_\alpha) + \mathrm{val}_{X_\alpha}(a_\gamma) \tag{4.8.2}$$

for $\gamma = \varphi_\beta, g_\beta$ where $\beta \neq \alpha$. Hence, by (4.8.1) we obtain

$$p\mathrm{val}_{X_\alpha}(a_\gamma) + \mathrm{val}_{X_\alpha}(a_\alpha) = \mathrm{val}_{X_\alpha}(a_\alpha) + \mathrm{val}_{X_\alpha}(a_\gamma). \tag{4.8.3}$$

Now (4.8.3) immediately yields $\mathrm{val}_{X_\alpha}(a_\gamma) = 0$ as desired. □

Lemma 4.10. *There exists an integer $k = k(D) > 0$ such that for any $\gamma \in \{\varphi_\beta \mid \beta \in \Delta \setminus \{\alpha\}\} \cup G_{L,\Delta}$, we have*

$$X^k_\alpha D^+_{\overline{\alpha}} \subseteq E^+_{L,\Delta}\gamma(D^+_{\overline{\alpha}}) \subseteq E^+_{\overline{\alpha}}\gamma(D^+_{\overline{\alpha}}).$$

Proof. The proof for the first inclusion relation follows exactly as in [50, Lemma 2.9]. The second inclusion relation is obvious as $E^+_{L,\Delta} \subseteq E^+_{\overline{\alpha}}$ by definition. □

Let us now define

$$D^{+*}_{\overline{\alpha}} := \bigcap_\gamma E^+_{\overline{\alpha}}\gamma(D^+_{\overline{\alpha}}),$$

where γ runs on the operators φ_β for $\beta \neq \alpha$ and on $G_{L,\Delta}$. $D^{+*}_{\overline{\alpha}}$ is finitely generated over $E^+_{\overline{\alpha}}$ as it is contained in $D^+_{\overline{\alpha}}$ and $E^+_{\overline{\alpha}}$ is noetherian. By Lemma 4.10, we conclude that $X^k_\alpha D^+_{\overline{\alpha}} \subseteq D^{+*}_{\overline{\alpha}}$ for some integer $k = k(D) > 0$. In particular, $D = D^{+*}_{\overline{\alpha}}[X^{-1}_\alpha]$.

Proposition 4.11. *$D^{+*}_{\overline{\alpha}}$ is an étale module over $E^{+}_{\overline{\alpha}}$, i.e. the maps*

$$\mathrm{id}\otimes\gamma\colon \gamma^{*}D^{+*}_{\overline{\alpha}} = E^{+}_{\overline{\alpha}} \bigotimes_{E^{+}_{\overline{\alpha}},\gamma} D^{+*}_{\overline{\alpha}} \longrightarrow D^{+*}_{\overline{\alpha}}$$

are bijective for all $\gamma\in\{\varphi_\beta \mid \beta\in\Delta\setminus\{\alpha\}\}\cup G_{L,\Delta}$.

Proof. The only thing we need to check in order for the third author's arguments in [50, Proposition 2.10] to work is that $E^{+}_{\overline{\alpha}}$ (resp. $E_{L,\Delta}$, resp. E^{+}_{Δ}) is a finite free module over $\gamma(E^{+}_{\overline{\alpha}})$ (resp. over $\gamma(E_{L,\Delta})$, resp. over $\gamma(E^{+}_{L,\Delta})$). This is already true if $\gamma=\varphi_\beta$ because the action of φ_β on the variables is exactly the same as the third author's arguments. If $\gamma\in G_{L,\Delta}$ then this is automatic since $G_{L,\Delta}$ is a group, so the action of γ is bijective. The rest of the argument in the proof follows exactly as in [50, Proposition 2.10]. □

Lemma 4.12. *There exists a finitely generated $E^{+}_{L,\Delta}$-submodule $D_0\subset D^{+*}_{\overline{\alpha}}$ such that $D_0\subseteq E^{+}_{L,\Delta}\varphi_{\overline{\alpha}}(D_0)$ and $D^{+*}_{\overline{\alpha}}=D_0[X^{-1}_{\Delta\setminus\{\alpha\}}]$, where $\varphi_{\overline{\alpha}}:=\prod_{\beta\in\Delta\setminus\{\alpha\}}\varphi_\beta$. Moreover, we have*

$$D^{+*}_{\overline{\alpha}}=\bigcup_{r\geq 0}E^{+}_{L,\Delta}\varphi^{r}_{\overline{\alpha}}(X^{-1}_{\Delta\setminus\{\alpha\}}D_0).$$

Proof. The proof is exactly the same as [50, Lemma 2.11]. □

5. The equivalence of categories for $\mathbb{F}_p$-representations

5.1. **The functor $\mathbb{D}$.** Let $\mathcal{H}_{L,\Delta}:=\prod_{\alpha\in\Delta}\mathcal{H}_{L_\alpha}$, where $\mathcal{H}_{L_\alpha}:=\mathrm{Gal}(\overline{K}_\alpha/L_{\alpha,\infty})$ for each $\alpha\in\Delta$. In case $L_\alpha=K_\alpha$ we omit the subscript K from the notation, ie. we put $\mathcal{H}_\Delta:=\prod_{\alpha\in\Delta}\mathcal{H}_\alpha$ where $\mathcal{H}_\alpha:=\mathrm{Gal}(\overline{K}_\alpha/K_{\alpha,\infty})$. Recall the field of norm E_α of $K_{\alpha,\infty}$ is isomorphic to $E_\alpha\cong k_\alpha((\overline{\pi}_\alpha))$. We already know by [1, Corollary 6.4] that $(E^{\mathrm{sep}}_\alpha)^{\mathcal{H}_\alpha}\cong E_\alpha$. For each $\alpha\in\Delta$, consider a finite separable extension E'_α of E_α together with the natural Frobenius $\varphi_\alpha: E'_\alpha\longrightarrow E'_\alpha$. The structure theorem for local fields of equal characteristic shows that $E'_\alpha\cong k_\alpha((X'_\alpha))$, where k_α is a finite extension over k_α. The field k_α is also the residue field of E'_α and X'_α is a uniformizer of E'_α. We denote by $E'^{+}_\alpha\cong k_\alpha[\![X'_\alpha]\!]$ in E'_α. As in [50, Section 3.1], we equip the tensor product

$$E'_{\Delta,\circ}:=\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}E'_\alpha$$

with a norm $|\cdot|_{\mathrm{prod}}$ by the formula

$$|c|_{\mathrm{prod}}:=\inf\left(\max_i\left(\prod_{\alpha\in\Delta}|c_{\alpha,i}|_\alpha\right)\middle|\, c=\sum_{i=1}^{n}\bigotimes_{\alpha\in\Delta}c_{\alpha,i}\right).$$

Note that the restriction of $|\cdot|_{\mathrm{prod}}$ to the subring $E'^{+}_{\Delta,\circ}:=\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}E'^{+}_\alpha$ induces the valuation with respect to the augmentation ideal $\mathrm{Ker}(E'^{+}_{\Delta,\circ}\twoheadrightarrow\bigotimes_{\alpha\in\Delta,\mathbb{F}_p}k_\alpha)$. Note that $\bigotimes_{\alpha\in\Delta,k_\alpha}k_\alpha$ is not a domain, and hence $|\cdot|_{\mathrm{prod}}$ is not multiplicative in general. However, it is submultiplicative. Following [50], we define E'^{+}_{Δ} as the completion of $E'^{+}_{\Delta,\circ}$ with respect to $|\cdot|_{\mathrm{prod}}$ and put $E'_\Delta:=E'^{+}_\Delta[1/X_\Delta]$. This ring E'_Δ is not complete with respect to $|\cdot|_{\mathrm{prod}}$ (unless $|\Delta|=1$).

Further, φ_α acts on $E'^{+}_{\Delta,\circ}$ (and on $E'_{\Delta,\circ}$) by the Frobenius on the component E'_α in E'_Δ and by the identity on all the other components in E'_β for $\beta\in\Delta\backslash\{\alpha\}$. This action is continuous in the norm $|\cdot|_{\mathrm{prod}}$ and therefore extends to the completion E'^{+}_Δ and the localization E'_Δ.

We define the multivariable analogue of E^{sep} as

$$E^{\mathrm{sep}}_\Delta := \varinjlim_{E_\alpha\leqslant E'_\alpha\leqslant E^{sep}_\alpha,\forall\alpha\in\Delta} E'_\Delta.$$

For any subset $\Delta'\subseteq\Delta$, one can define the similar notions $E'^{+}_{\Delta'}$, $E'_{\Delta'}$ and $E^{\mathrm{sep}}_{\Delta'}$ with Δ replaced by Δ'. We equip E^{sep}_Δ with the relative Frobenii φ_α for each $\alpha\in\Delta$ and the absolute Frobenius φ_s defined above on each E'_Δ. Further E^{sep}_Δ admits a Galois action of the Galois group $\mathcal{G}_\Delta$.

With respect to the ring E'_Δ, we have the following alternative characterization.

Lemma 5.1. *Put $\Delta=\{\alpha_1,\cdots,\alpha_n\}$. We have*

$$E'_\Delta\cong E'_{\alpha_1}\bigotimes_{E_{\alpha_1}}\left(E'_{\alpha_2}\bigotimes_{E_{\alpha_2}}\left(\cdots\left(E'_{\alpha_n}\bigotimes_{E_{\alpha_n}}E_\Delta\right)\right)\right).$$

Proof. The proof of [50, Lemma 3.2] also works in our imperfect case. □

Proposition 5.2. *For a collection of finite separable extensions E'_α/E_α $\alpha\in\Delta$ we let $\mathcal{H}'_\Delta:=\prod_{\alpha\in\Delta}\mathcal{H}'_\alpha$, where $\mathcal{H}'_\alpha=\mathrm{Gal}(E^{\mathrm{sep}}_\alpha/E'_\alpha)$. Then we have $(E^{\mathrm{sep}}_\Delta)^{\mathcal{H}'_\Delta}=E'_\Delta$. In particular, we have $(E^{\mathrm{sep}}_\Delta)^{\mathcal{H}_{L,\Delta}}=E_{L,\Delta}$.*

Proof. Since X_Δ is $\mathcal{H}'_\Delta$-invariant and $\varinjlim$ can be interchanged with taking $\mathcal{H}'_\Delta$-invariants, it suffices to show that whenever

$$E_\alpha=k_\alpha((X_\alpha))\leqslant E'_\alpha=k_{\alpha'}((X'_\alpha))\leqslant E''_\alpha=k_{\alpha''}((X''_\alpha))$$

is a sequence of finite separable extensions for each $\alpha\in\Delta$ such that E''_α/E'_α is Galois, then we have $(E''^{+}_\Delta)^{\mathcal{H}'_\Delta}=E'^{+}_\Delta$. The containment $E'^{+}_\Delta\subseteq(E''^{+}_\Delta)^{\mathcal{H}'_\Delta}$ is clear. For the converse, we will prove by induction on $|\Delta|$. It should be remarked that the ideal $\mathcal{M}_\alpha\triangleleft E''^{+}_\Delta$ generated by X''_α is invariant under the action of $\mathcal{H}'_\Delta$ for any fixed $\alpha\in\Delta$. Moreover, for any integer $k\geq 1$, the ring $E''^{+}_\alpha/\mathcal{M}^k_\alpha$ is finite dimensional over k_α. Therefore the image of $(E''^{+}_\Delta)^{\mathcal{H}'_\Delta}$ under the quotient map $E''^{+}_\Delta\twoheadrightarrow E''^{+}_\Delta/\mathcal{M}^k_\alpha$ is contained in

$$\begin{aligned}\left(E''^{+}_\Delta/\mathcal{M}^k_\alpha\right)^{\mathcal{H}'_\Delta}\subseteq\left(E''^{+}_\Delta/\mathcal{M}^k_\alpha\right)^{\mathcal{H}'_{\Delta\backslash\{\alpha\}}}&=\left(E''^{+}_{\Delta\backslash\{\alpha\}}\bigotimes_{\mathbb{F}_p}\left(E''^{+}_\alpha/\mathcal{M}^k_\alpha\right)\right)^{\mathcal{H}'_{\Delta\backslash\{\alpha\}}}\\&=\left(E''^{+}_{\Delta\backslash\{\alpha\}}\right)^{\mathcal{H}'_{\Delta\backslash\{\alpha\}}}\bigotimes_{\mathbb{F}_p}(E''^{+}_\alpha/\mathcal{M}^k_\alpha)\\&=E'^{+}_{\Delta\backslash\{\alpha\}}\bigotimes_{\mathbb{F}_p}(E''^{+}_\alpha/\mathcal{M}^k_\alpha)\end{aligned}\tag{5.2.1}$$

by induction. Note that the second equality in (5.2.1) follows from the following fact.

Fact. If A and B are k-vector spaces with a G-action such that B is finite dimensional over k and B has trivial G-action, then

$$(A \bigotimes_k B)^G \cong (A \bigotimes_k k^r)^G \cong (A^r)^G \cong A^G \bigotimes_k k^r \cong A^G \bigotimes_k B,$$

where $B \cong k^r$.

By taking inductive limits of finite dimensional vector spaces and as inductive limit commute with these operations, the assumption that B is finite dimensional over k can be removed from this fact.

Let us continue our proof. Taking the projective limit of $(E_\Delta^{\prime\prime+}/\mathcal{M}_\alpha^k)^{\mathcal{H}'_\Delta}$ with respect to $k \geq 1$, we deduce that $(E_\Delta^{\prime\prime+})^{\mathcal{H}'_\Delta}$ is contained in the power series ring

$$\left(k_{\alpha''} \bigotimes_{\mathbb{F}_p} \bigotimes_{\beta \in \Delta \setminus \{\alpha\}, \mathbb{F}_p} k_{\beta'} \right) [\![X''_\alpha, X'_\beta \,|\, \beta \in \Delta \setminus \{\alpha\}]\!] \subseteq E_\Delta^{\prime\prime+}.$$

Indeed, even though projective limits do not commute in general with taking $\mathcal{H}'_\Delta$-invariants, this inclusion is automatic. Now using the action of $\mathcal{H}'_\alpha$ in a similar argument as above (reducing modulo the k-th power of the ideal generated by all the $X'_\beta, \beta \in \Delta \setminus \{\alpha\}$ for all $k \geq 1$) we deduce the statement. □

We define the subring $E_{\Delta,\circ}^{\mathrm{sep}} \cong \bigotimes_{\alpha \in \Delta, \mathbb{F}_p} E_\alpha^{\mathrm{sep}}$ in E_Δ^{sep} to be the inductive limits of $E'_{\Delta,\circ} \subseteq E'_\Delta$ where E'_α runs through the finite separable extensions of E_α for each $\alpha \in \Delta$.

As in section 2. assume L_α/K_α is a finite totally ramified extension for each $\alpha \in \Delta$, ie. L_α also has k_α as residue field. Put $E_{L,\Delta} := (E_\Delta^{\mathrm{sep}})^{\mathcal{H}_{L,\Delta}}$ where $\mathcal{H}_{L,\Delta} := \prod_{\alpha \in \Delta} \mathcal{H}_{L_\alpha}$. As in the case $L = K$ a $(\varphi_\Delta, G_{L,\Delta})$-module over $E_{L,\Delta}$ is a finitely generated free module D over $E_{L,\Delta}$ together with commuting semilinear actions of φ_α and the Galois groups G_{L_α} for all $\alpha \in \Delta$. By an étale $(\varphi_\Delta, G_{L,\Delta})$-module over $E_{L,\Delta}$, we mean a $(\varphi_\Delta, G_{L,\Delta})$-module D such that the maps

$$\mathrm{id} \otimes \varphi_\alpha : \varphi_\alpha^* D := E_{L,\Delta} \bigotimes_{E_{L,\Delta}, \varphi_\alpha} D \longrightarrow D,$$

are isomorphisms for all $\alpha \in \Delta$.

Now let V be a finite dimensional representation of the group $\mathcal{G}_{L,\Delta}$ over $\mathbb{F}_p$. The basechange $E_\Delta^{\mathrm{sep}} \bigotimes_{\mathbb{F}_p} V$ is equipped with the diagonal semilinear action of $\mathcal{G}_{L,\Delta}$ and with the partial Frobenii $\varphi_\alpha\,(\alpha \in \Delta)$. These all commute with each other. We define the functor $\mathbb{D}$ as in [50]

$$\mathbb{D}(V) := (E_\Delta^{\mathrm{sep}} \bigotimes_{\mathbb{F}_p} V)^{\mathcal{H}_{L,\Delta}}.$$

By Proposition 5.2, $\mathbb{D}(V)$ is a module over E_Δ which inherits the action of $\varphi_\alpha\,(\alpha \in \Delta)$, and the Galois group $\mathcal{G}_{L,\Delta}$ on $E_\Delta^{\mathrm{sep}} \bigotimes_{\mathbb{F}_p} V$. Further, the action of $\mathcal{G}_{L,\Delta}$ factors through its quotient $G_{L,\Delta} = \mathcal{G}_{L,\Delta}/\mathcal{H}_{L,\Delta}$. We denote the category of étale $(\varphi_\Delta, G_{L,\Delta})$-modules over $E_{L,\Delta}$ by $\mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$. One key Lemma for us is the following.

Lemma 5.3. *The E_Δ^{sep}-module $E_\Delta^{\mathrm{sep}} \bigotimes_{\mathbb{F}_p} V$ admits a basis consisting of elements fixed by $\mathcal{H}_{L,\Delta}$.*

Proof. The same proof given in [50, Lemma 3.4] exactly works here. □

Lemma 5.4. *We have* $(E_\Delta^{\mathrm{sep}})^\times \cap E_{L,\Delta} = E_{L,\Delta}^\times$.

Proof. Let u be an arbitrary element in $(E_\Delta^{\mathrm{sep}})^\times \cap E_{L,\Delta}$. Since u is invariant under the action of $\mathcal{H}_{L,\Delta}$, so is its inverse u^{-1}. And hence it also lies in $E_{L,\Delta}$ by Proposition 5.2. □

Lemma 5.5. *We have* $\bigcap_{\alpha\in\Delta} (E_\Delta^{\mathrm{sep}})^{\varphi_\alpha=\mathrm{id}} = \mathbb{F}_p$.

Proof. The containment $\mathbb{F}_p \subseteq \bigcap_{\alpha\in\Delta} (E_\Delta^{\mathrm{sep}})^{\varphi_\alpha=\mathrm{id}}$ is obvious. On the other hand, let $u \in E_\Delta^{\mathrm{sep}}$ be an arbitrary element such that $\varphi_\alpha(u) = u$ for all $\alpha \in \Delta$. We also have $u^p = \varphi_s(u) = u$ as $\varphi_s = \prod_{\alpha\in\Delta} \varphi_\alpha$ is the absolute Frobenius on E_Δ^{sep}. Since E_Δ^{sep} is defined to be an inductive limit, u lies in $E'_\Delta \cong \left(\bigotimes_{\alpha\in\Delta,\mathbb{F}_p} k'_\alpha\right) [\![X'_\alpha \mid \alpha\in\Delta]\!][X_\Delta^{-1}]$ for some collection $E'_\alpha = k'_\alpha((X'_\alpha))$ $(\alpha\in\Delta)$ of finite separable extensions of E_α.

Since k'_α is a finite separable extension of k_α, it is a finitely generated field extension of $\mathbb{F}_p$. Therefore by Lemma 2.3 $k'_\Delta := \bigotimes_{\alpha\in\Delta,\mathbb{F}_p} k'_\alpha$ is reduced. This implies that for each $\alpha \in \Delta$ the norm $|\cdot|_\alpha$ on $E'_\Delta = k'_\Delta[\![X'_\alpha \mid \alpha\in\Delta]\!][X_\Delta^{-1}]$ defined by the X'_α-adic valuation is power-multiplicative as the powers of the leading coefficient (with respect to the X'_α-degree) of an element cannot vanish. Therefore, we have $|u^p|_\alpha = |u|_\alpha^p$ for all $\alpha\in\Delta$. We deduce $|u|_\alpha = 1$ unless $u = 0$. In particular, u lies in $E'^{+}_\Delta = k'_\Delta[\![X'_\alpha \mid \alpha\in\Delta]\!]$. The constant term $u_0 \in k'_\Delta$ also satisfies $\varphi_\alpha(u_0) = u_0$ for all $\alpha\in\Delta$. Now, k'_Δ is an infinite dimensional vector space over $\mathbb{F}_p$. For a fixed $\alpha\in\Delta$, we can choose elements of an $\mathbb{F}_p$-basis $d_1, \cdots, d_n$ of $\bigotimes_{\beta\in\Delta\setminus\{\alpha\},\mathbb{F}_p} k'_\beta$ such that $u_0 = \sum_{i=1}^n c_i \otimes d_i$ with $c_i \in k'_\alpha$. This decomposition is unique and we compute

$$\sum_{i=1}^n c_i \otimes d_i = u_0 = \varphi_\alpha(u_0) = \sum_{i=1}^n c_i^p \otimes d_i.$$

We conclude $c_i = c_i^p$. But, (when $|\Delta| = 1$) by [46, Theorem 2.1.3] and [46, Equation 2.1.5 and Equation 2.1.6], we know that $(E_\alpha^{\mathrm{sep}})^{\varphi_\alpha=\mathrm{id}} = \mathbb{F}_p$. Therefore $c_i \in \mathbb{F}_p$ for all $1 \leqslant i \leqslant n$. It follows by induction on $|\Delta|$ that u_0 lies in $\mathbb{F}_p$. Now $u - u_0$ is also fixed by each φ_α $(\alpha\in\Delta)$ and φ_s, but we have $|u-u_0|_{\mathrm{prod}} < 1$. This implies by the discussion above that $u = u_0$ is in $\mathbb{F}_p$ as desired. □

Proposition 5.6. $\mathbb{D}(V)$ *is an étale* $(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$*-module over* $E_{L,\Delta}$ *of rank* $d := \dim_{\mathbb{F}_p} V$. *Moreover, we have*

$$E_\Delta^{\mathrm{sep}} \bigotimes_{E_{L,\Delta}} \mathbb{D}(V) \cong E_\Delta^{\mathrm{sep}} \bigotimes_{\mathbb{F}_p} V,$$

and

$$V = \bigcap_{\alpha\in\Delta} \left(E_\Delta^{\mathrm{sep}} \bigotimes_{E_{L,\Delta}} \mathbb{D}(V)\right)^{\varphi_\alpha=\mathrm{id}}.$$

Proof. By Proposition 5.2 and Lemma 5.3 we can say that $\mathbb{D}(V)$ is a free module of rank d over $E_{L,\Delta}$. Moreover, the matrix of φ_α in any basis of $\mathbb{D}(V)$ is invertible in E_Δ^{sep}, therefore also in $E_{L,\Delta}$ by Lemma 5.4. So the action of $(\varphi_\Delta, G_{L,\Delta})$ on $\mathbb{D}(V)$ is étale. The last statement is a direct consequence of Lemmas 5.3 and 5.5. □

Lemma 5.7. *For objects V, V_1, V_2 in $\mathrm{Rep}_{\mathbb{F}_p}(\mathcal{G}_{L,\Delta})$, we have $\mathbb{D}(V_1 \bigotimes_{\mathbb{F}_p} V_2) \cong \mathbb{D}(V_1) \bigotimes_{E_{L,\Delta}} \mathbb{D}(V_2)$ and $\mathbb{D}(V^*) \cong \mathbb{D}(V)^*$.*

Proof. The proof of [50, Lemma 3.8] works in our imperfect case. □

Theorem 5.8. *$\mathbb{D}$ is a fully faithful tensor functor from the category $\mathrm{Rep}_{\mathbb{F}_p}(\mathcal{G}_{L,\Delta})$ to the category $\mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$.*

Proof. Let $f : V_1 \longrightarrow V_2$ be a nonzero morphism in $\mathrm{Rep}_{\mathbb{F}_p}(\mathcal{G}_{L,\Delta})$. Then the E_Δ^{sep}-linear map $\mathrm{id} \bigotimes f : E_\Delta^{\mathrm{sep}} \bigotimes_{\mathbb{F}_p} V_1 \longrightarrow E_\Delta^{\mathrm{sep}} \bigotimes_{\mathbb{F}_p} V_2$ is also nonzero. By Proposition 5.6 we assert that $\mathbb{D}(f) \neq 0$, and therefore the faithfulness.

Now let V_1 and V_2 be arbitrary objects in $\mathrm{Rep}_{\mathbb{F}_p}(\mathcal{G}_{L,\Delta})$ and $\theta : \mathbb{D}(V_1) \longrightarrow \mathbb{D}(V_2)$ be a morphism in $\mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$. Then by Proposition 5.6, we obtain a $\mathcal{G}_{L,\Delta}$-equivariant $\mathbb{F}_p$-linear map

$$f : V_1 = \bigcap_{\alpha\in\Delta} \left(E_\Delta^{\mathrm{sep}} \bigotimes_{E_{L,\Delta}} \mathbb{D}(V_1) \right)^{\varphi_\alpha = \mathrm{id}} \longrightarrow \bigcap_{\alpha\in\Delta} \left(E_\Delta^{\mathrm{sep}} \bigotimes_{E_{L,\Delta}} \mathbb{D}(V_2) \right)^{\varphi_\alpha = \mathrm{id}} = V_2$$

induced by θ for which we have $\theta = \mathbb{D}(f)$. Therefore $\mathbb{D}$ is full. The compatibility with tensor product follows from Lemma 5.7. □

5.2. **The functor $\mathbb{V}$.** In the following, we define the functor $\mathbb{V}$ and show that it is a quasi-inverse of $\mathbb{D}$. This, in turn, implies that the functor $\mathbb{D}$ is essentially surjective. Let $D \in \mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$. It comes with a natural semilinear action of φ_α ($\alpha \in \Delta$) and the Galois group $G_{L,\Delta}$. We define

$$\mathbb{V}(D) := \bigcap_{\alpha\in\Delta} \left(E_\Delta^{\mathrm{sep}} \bigotimes_{E_{L,\Delta}} D \right)^{\varphi_\alpha = \mathrm{id}}.$$

$\mathbb{V}(D)$ is a—a priori not necessarily finite dimensional—representation of $\mathcal{G}_{L,\Delta}$ over $\mathbb{F}_p$.

Lemma 5.9. *For any integer $r > 0$ we have*

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}} \otimes_{E^+_{L,\Delta\setminus\{\alpha\}}} E^+_{L,\Delta}/(X_\alpha^r) \right)^{\varphi_\beta = \mathrm{id}} = k_\alpha[X_\alpha]/(X_\alpha^r).$$

Proof. First of all, we have

$$E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}} \otimes_{E^+_{\Delta\setminus\{\alpha\}}} E^+_{L,\Delta}/(X_\alpha^r) = \varinjlim E'_{\beta_1} \bigotimes_{E_{\beta_1}} \left(E'_{\beta_2} \bigotimes_{E_{\beta_2}} \left(\cdots \left(E'_{\beta_{n-1}} \bigotimes_{E_{\beta_{n-1}}} E_{L,\Delta}/(X_\alpha^r) \right) \right) \right).$$

where $\Delta \setminus \{\alpha\} = \{\beta_1, \ldots, \beta_{n-1}\}$. As in Lemma 5.5 if $\varphi_\beta(u) = u$ for some element $u \in E'_{\Delta\setminus\{\alpha\}} \otimes_{E^+_{\Delta\setminus\{\alpha\}}} E^+_{L,\Delta}/(X_\alpha^r)$ then by Lemma 2.3 we must have $p\mathrm{val}_{X'_\beta}(u) = \mathrm{val}_{X'_\beta}(u)$ showing

$\mathrm{val}_{X_{\beta'}}(u) = 0$. The constant term (with respect to the variables X_β, $\beta \in \Delta \setminus \{\alpha\}$) is an element in $k_{\Delta\setminus\{\alpha\}} \otimes_{\mathbb{F}_p} k_\alpha[X_\alpha]/(X_\alpha^r)$. So the statement follows the same way as in the proof of Lemma 5.5 noting

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(k_{\Delta\setminus\{\alpha\}}\right)^{\varphi_\beta=\mathrm{id}} = \mathbb{F}_p$$

and φ_β acts trivially on $k_\alpha[X_\alpha]/(X_\alpha^r)$. □

Lemma 5.10. *For any integer $r > 0$ and finitely generated $E_{\overline{\alpha}}^+/(X_\alpha^r)$-module M we have an identification*

$$E_{\Delta\setminus\{\alpha\}}^{\mathrm{sep}}[X_\alpha]/(X_\alpha^r) \bigotimes_{E_{\overline{\alpha}}^+/(X_\alpha^r)} M \cong E_{\Delta\setminus\{\alpha\}}^{\mathrm{sep}} \bigotimes_{E_{\Delta\setminus\{\alpha\}}} M.$$

Proof. This follows from the isomorphism $E_{\overline{\alpha}}^+/(X_\alpha^r) \cong E_{\Delta\setminus\{\alpha\}} \otimes_{\mathbb{F}_p} k_\alpha[X_\alpha]/(X_\alpha^r)$. □

For a subset $\Delta' \subseteq \Delta$, we put $E_{\Delta'}^{\mathrm{sep}+} := \varinjlim E_{\Delta'}'^{+}$ so we have $E_{\Delta'}^{\mathrm{sep}} = E_{\Delta'}^{\mathrm{sep}+}[X_{\Delta'}^{-1}]$, where $X_{\Delta'} := \prod_{\alpha\in\Delta'} X_\alpha$.

The proofs of the following two Lemmas follow exactly as in [50, Lemma 3.13] and [50, Lemma 3.14] without any change.

Lemma 5.11. *$E_{\Delta'}^{\mathrm{sep}}$ (resp. $E_{\Delta'}^{\mathrm{sep}+}$) is flat as a module over $E_{\Delta'}$ (resp. over $E_{\Delta'}^+$) for all $\Delta' \subseteq \Delta$.*

Lemma 5.12. *We have $\left(E_{\Delta\setminus\{\alpha\}}^{\mathrm{sep}+}[\![X_\alpha]\!][X_\Delta^{-1}]\right)^{\mathcal{H}_{\Delta\setminus\{\alpha\}}} = E_{L,\Delta}$.*

Our main result in this section is the following

Theorem 5.13. *The functors $\mathbb{D}$ and $\mathbb{V}$ are quasi-inverse equivalences of categories between $\mathrm{Rep}_{\mathbb{F}_p}(\mathcal{G}_{L,\Delta})$ and $\mathcal{D}^{\mathrm{et}}(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$.*

Proof. The proof is essentially the same as that of [50, Theorem 3.15]. Instead of repeating the proof, we recall the strategy and just point out, what the changes should be adapted to making the argument work in our imperfect residue field case.

Step1. *Reducing the statement to the essential surjectivity of $\mathbb{D}$.* By Theorem 5.8 $\mathbb{D}$ is fully faithful and by Proposition 5.6 we have $\mathbb{V}\circ\mathbb{D}(V) \cong V$ for any object V in $\mathrm{Rep}_{\mathbb{F}_p}(\mathcal{G}_{L,\Delta})$. So we are reduced to showing that $\mathbb{D}$ is essentially surjective. This is done by induction on $|\Delta|$. The case of $|\Delta| = 1$ is due to Scholl [46] and Andreatta [1, Theorem 7.11]. Suppose that $|\Delta| > 1$, fix $\alpha \in \Delta$, and pick an object D in $\mathcal{D}^{et}(\varphi_\Delta, G_{L,\Delta}, E_{L,\Delta})$.

Step 2. *The goal here is to trivialize the $(\varphi_{\overline{\alpha}}, \varphi_\beta)$-action ($\beta \in \Delta\setminus\{\alpha\}$) on $D_{\overline{\alpha}}^{+*}/X_\alpha^r D_{\overline{\alpha}}^{+*}$ uniformly in r by tensoring up with $E_{\Delta\setminus\{\alpha\}}^{sep}$.* The place which we need to modify is the

Equation (4) before [50, Lemma 3.17]. In our imperfect case, it should be reformulated as

$$\begin{aligned} E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r) \bigotimes_{\mathbb{F}_p[X_\alpha]/(X_\alpha^r)} \bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r) \bigotimes_{E^+_{\overline{\alpha}}/(X_\alpha^r)} D^{+*}_{\overline{\alpha},r}\right)^{\varphi_{\overline{\alpha}}=\mathrm{id},\ \varphi_\beta=\mathrm{id}} \\ \xrightarrow{\sim} E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r) \bigotimes_{E^+_{\overline{\alpha}}/(X_\alpha^r)} D^{+*}_{\overline{\alpha},r} \\ \cong E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r) \bigotimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}}. \end{aligned}$$

Lemma 3.17 in [50] should be changed into: there exists a finitely generated $E^+_{L,\Delta}$-submodule $M \leqslant D^{+*}_{\overline{\alpha}}$ such that

$$\bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r) \bigotimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}}\right)^{\varphi_{\overline{\alpha}}=\mathrm{id},\varphi_\beta=\mathrm{id}}$$

is contained in the image of the map

$$\begin{aligned} E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r) \bigotimes_{E^+_{L,\Delta}} M \longrightarrow E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r) \bigotimes_{E^+_{L,\Delta}} D^{+*}_{\overline{\alpha}} \\ \cong E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r) \bigotimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}} \end{aligned}$$

induced by the inclusion $M \leqslant D^{+*}_{\overline{\alpha}}$ for all $r > 0$.

Note that the intersection in the above-mentioned is not only over $\varphi_\beta = \mathrm{id}$ $(\beta \in \Delta\setminus\{\alpha\})$ but also over $\varphi_{\overline{\alpha}} = \mathrm{id}$, noting that $\varphi_{\overline{\alpha}}$ is the absolute Frobenius of $E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}$. The reason why we need to make this change is because it is coherent with our Lemma 5.5 and also in the proof of [50, Lemma 3.17] (where the third author use the fact that $\varphi_{\overline{\alpha}}^{l_r}(x) = x$).

Step 3. *The goal of this step is to show the following compatibility of our construction with projective limits with respect to r.* In our imperfect residue field case, Lemma 3.18 in [50] should be changed into: we have

$$\varprojlim_r \left(E^{\mathrm{sep}+}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r) \bigotimes_{E^+_{L,\Delta}} M \right) \cong E^{\mathrm{sep}+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \bigotimes_{E^+_{L,\Delta}} M,$$

$$\varprojlim_r \left(E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r) \bigotimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}} \right) \cong E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \bigotimes_{E^+_{\overline{\alpha}}} D^{+*}_{\overline{\alpha}},$$

and

$$\varprojlim_r \left(E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r) \bigotimes_{\mathbb{F}_p[X_\alpha]/(X_\alpha^r)} \bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[X_\alpha]/(X_\alpha^r) \bigotimes_{E^+_{\overline{\alpha}}/(X_\alpha^r)} D^{+*}_{\overline{\alpha},r} \right)^{\varphi_{\overline{\alpha}}=\mathrm{id},\varphi_\beta=\mathrm{id}} \right)$$

$$\cong E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \bigotimes_{\mathbb{F}_p[\![X_\alpha]\!]} \bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!] \bigotimes_{E_{L,\Delta}} D \right)^{\varphi_{\overline{\alpha}}=\mathrm{id},\varphi_\beta=\mathrm{id}}.$$

Everything else including the proof of [50, Lemma 3.18] remains the same.

Step 4. *Step 4. The goal here is to obtain a $(\varphi_\alpha, G_\alpha)$-module D_α over E_α (by trivializing the action of $(\varphi_{\overline{\alpha}}, \varphi_\beta), \beta \in \Delta\setminus\{\alpha\}$) which is at the same time a linear representation of the group $\mathcal{G}_{L,\Delta\setminus\{\alpha\}}$.* In the Step 4 of the third author's proof (before [50, Lemma 3.19]), the corresponding D_α in our imperfect residue field case should be defined as

$$D_\alpha := \bigcap_{\beta\in\Delta\setminus\{\alpha\}} \left(E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}((X_\alpha)) \bigotimes_{E_{L,\Delta}} D \right)^{\varphi_{\overline{\alpha}}=\mathrm{id},\varphi_\beta=\mathrm{id}},$$

which is contained in the image of the map

$$E^{\mathrm{sep}+}_{\Delta\setminus\{\alpha\}}[\![X_\alpha]\!][X_\alpha^{-1}] \bigotimes_{E_{L,\Delta}} D \hookrightarrow E^{\mathrm{sep}}_{\Delta\setminus\{\alpha\}}((X_\alpha)) \bigotimes_{E_{L,\Delta}} D.$$

Then D_α is an $\mathbb{F}_p((X_\alpha))$ vector space.

Step 5. *The last step is to show the essential surjectivity of* $\mathbb{D}$. Lemma 3.19 of [50] exactly remains the same, including its proof, and so does the third author's Step 5 (cf. see after the proof of [50, Lemma 3.19]). We do not need to change anything here. □

Corollary 5.14. *Any object D in $\mathcal{D}^{\mathrm{et}}(\varphi_\Delta, \varphi_s, G_{L,\Delta}, E_{L,\Delta})$ is a free module over $E_{L,\Delta}$.*

Proof. By Theorem 5.13 we know that $\mathbb{D}$ is essentially surjective. The Corollary follows by noting that any étale module in the image of the functor $\mathbb{D}$ is free as a module over $E_{L,\Delta}$ by construction. □

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